Conditionalization is simple and intuitive as a rule of rational belief change in response to evidence. But sometimes its can be a little tricky to apply.

> **Monty Hall**. On the Monty Hall game show a contestant is shown three doors. Behind two doors are goat. Behind a third is a brand new car. Once you pick a door, before opening it, Monty Hall opens one *other* door to reveal a goat. Then he gives you the option of switching from the door you originally chose to the other unopened door. Should you switch?

Argument: Switching doesn't help! Monty's eliminated one door, but you knew he was going to do *that* no matter what. So you haven't learned anything new except that the car isn't behind door 3 (say). Conditionalizing on that information, we should believe it's 50% likely the car is behind each of the remaining doors.

This argument is (very subtly) fallacious. Why? We'll come back to this...

Bayes' Theorem

Recall the definition of conditional probability

$$P(d|e) = \frac{P(e \text{ and } d)}{P(e)}$$

Multiplying by P(e), we get a formula describing the probability of conjunctive events.

$$P(e \text{ and } d) = P(d|e)P(e)$$

Watch out. As always, P(e) must be non-zero for this to apply. Both equations can be useful: sometimes it's easier to settle conditional probabilities, sometimes conjunctive probabilities.

There's a special case of importance. Let's say that ***d is probabilistically independent of e*** if P(d|e)=P(d) (or if P(e) or P(d)=0). Intuitively: e doesn't bear on whether or not d holds. Then we have

$$P(e \text{ and } d) = P(d)P(e)$$

Consider: What's the chances that if you flip a coin three times, the first will come out heads and the next two tails? What are the chances that if you draw two cards at random from a deck, they'll both be aces?

Note that P(e and d) = P(d and e) = P(e|d)P(d). Let's substitute the right-hand side of this equation for the left hand side, as it appears in the definition of conditional probability. Then we arrive at what's known as ***Bayes' Theorem***:

$$P(d|e) = \frac{P(e|d)P(d)}{P(e)}$$

Again, watch out. *Both* P(e) and P(d) must be non-zero for this to apply. In addition to having a number of very useful applications, Bayes' Theorem has an important significance for the utility of conditionalization. More on that later. For now, more examples...

Example 1

10% of applicants, and 12% of admits to Pitt this year will be from Allegheny County. 50% of all applicants will be admitted. You learn of arbitrary applicant that she's from Allegheny. How likely is it she will be accepted?

Example 2

You live in a city where it rains very little: on average, 3.65 days a year. You're deciding whether to plan for a big outdoor party on Saturday, but when you turn on the TV, you see the weatherman has forecast rain. Historically the weatherman is pretty accurate: when it rained, he was right 90% of the time. When it didn't rain, he incorrectly predicted rain only 10% of the time. Using this information, you settle that the weatherman predicts rain about once out of every 10 days. How likely should you think it is to rain on Saturday?

Sources of Evidence *as* Evidence

Back to the Monty Hall problem. Let's use the new tools we've developed to think through what to do. The problem earlier was that you didn't just learn a goat was behind door 3, but that Monty revealed the goat was behind door 3. *Monty's choice is probabilistically related to where the goat is.* If you just conditionalized on the information that the car isn't behind door 3, you'd be missing out on important information that Monty's choice is giving you.

Let "door 1" be: the car is behind door 1.
Let "door 2" be: the car is behind door 2.
Let "reveals 3" be: Monty opens door 3.

$$
\begin{aligned}
P(\text{door 1}|\text{ reveals 3}) &= \frac{P(\text{reveals 3}|\text{door 1})\ P(\text{door 1})}{P(\text{reveals 3})} \\
&= \frac{1/2 \text{ x } 1/3}{1/2} \\
&= 1/3
\end{aligned}
$$

On the other hand...

$$
\begin{aligned}
P(\text{door 2}|\text{ reveals 3}) &= \frac{P(\text{reveals 3}|\text{door 2})\ P(\text{door 1})}{P(\text{reveals 3})} \\
&= \frac{1 \text{ x } 1/3}{1/2} \\
&= 2/3
\end{aligned}
$$

What happened? Note the only difference is in P(reveals 3|door n). This is the "added information" about the probabilistic relationship between Monty's action and the location of the goat that was missing before.

Compare this with a case where a friend, who knows nothing of where you pointed, randomly got a look back stage and gives you "insider information" that the goat isn't behind 3.

Here's another way to think about it. Right after you point here's how our priors should be assigned:

Car behind 2, Monty reveals 3: 1/3
Car behind 3, Monty reveals 2: 1/3
Car behind 1, Monty reveals 2: 1/6
Car behind 1, Monty reveals 3: 1/6

When you conditionalize, you rule out the second and third possibilities and "renormalize". But then the first option is twice as likely as the fourth.

Lesson: when you update on evidence, be sure to keep track of *all the evidence* (including any evidence about the nature, patterns, and reliability of the sources of your evidence!)