Handout 2
Uncertainty v. Risk
Probability I: Axioms

## $\mathbf{R}^{\begin{array}{l}\text { Decisions, Games \& } \\ \text { ATIONAL CHOICE }\end{array}}$

To know which decisions are rational, inevitably we need to think about how likely various states are to obtain. Consider two decisions: whether to drive to the store to get some food you crave, and (on a separate occasion) whether to eat some food you crave given that it's been exposed for hours to deadly toxins.

|  | No Accident if Drive | Accident if Drive |
| :--- | :---: | :---: |
| Drive to store | 10 | -100000 |
| Stay at home | 0 | 0 |


|  | Food Safe <br> despite Exposure | Food <br> Contaminated <br> during Exposure |
| :--- | :---: | :---: |
| Eat | 10 | -100000 |
| Don't | 0 | 0 |

Plausibly the answers to these two decision questions should be different. But the source of the different answers is the likelihood of the outcomes. This isn't yet reflected in our decision tables. We want to know: how to reason with likelihoods, what likelihood is, and then how best to integrate it into a theory of rational choice.

## Uncertainty v. Risk

There are two kinds of ignorance one can have about what state of the world obtains. You might not know which states obtain, but you have a very good idea how likely they are to obtain (as in, e.g., the outcome of the flip of a fair coin). Or, you might not know even that likelihood (e.g. you have an opportunity to invest in a friend's company, but without knowing what kind of company it is). These represent potentially different kinds of decision scenarios.

To the extent the outcome of a decision turns on states, where the agent has information, or firm beliefs, about the likelihood that those states obtain, it is a decision under risk.

To the extent the outcome of a decision turns on states, where the agent has no information, or firm beliefs, about the likelihood that those states obtain, it is a decision under uncertainty.

Some people claim that decisions under risk should be treated differently from decisions under uncertainty. (In particular, that rules appealing only to ordinal utilities might more sensibly apply to decisions under uncertainty, but not under risk).

But the distinction here isn't sharp. And how big is the difference here when it applies sharply?

## Axioms for Probability

It's common to see probabilities represented mathematically (e.g., by the weatherman). Mathematical representations help us state the "laws" that probabilities obey.

Start with a set of states: it rains tomorrow, it's sunny tomorrow, it rained yesterday, the Mets win the world series, it rains tomorrow and the Mets win the world series, and so on. Then we can imagine the probability of each and every one of these states occurring and encapsulate it in a function, that we'll write " P ". For any state s , $\mathrm{P}(\mathrm{s})$ is the probability of that state.

The "laws" of probability are, in one sense, incredibly simple. They are captured by the following three axioms.
(I) $\mathrm{P}(\mathrm{s}) \geq 0$ for all s .
(II) $\mathrm{P}(\mathrm{s})=1$ if s is necessary.
(III) $\mathrm{P}\left(\mathrm{s}\right.$ or $\left.\mathrm{s}^{\prime}\right)=\mathrm{P}(\mathrm{s})+\mathrm{P}\left(\mathrm{s}^{\prime}\right)$ if s and $\mathrm{s}^{\prime}$ are "mutually exclusive".
(if, necessarily, they cannot jointly hold)
Using only these axioms, we can prove everything there is to know about probability. For example:
(a) $\mathrm{P}(\mathrm{s}$ or not s$)=1$
(b) $\mathrm{P}($ not s$)=1-\mathrm{P}(\mathrm{s})$
(c) $\mathrm{P}(\mathrm{s})=0$ if s is necessarily false.
(d) $\mathrm{P}(\mathrm{s}) \leq 1$ for all s
(e) $\mathrm{P}(\mathrm{a}) \leq \mathrm{P}(\mathrm{b})$ if a entails b (whenever a is true, b must be true as well)
(f) $\mathrm{P}(\mathrm{a})=\mathrm{P}(\mathrm{b})$ if a is logically equivalent to b (a entails b and b entails a )

It will be very important to know facts about how probability relates to "logical operations" on states (states joined by "and", "or", "if...then..." etc."). The more general rule for how "or" is a bit more tricky.
(g) $\mathrm{P}(\mathrm{a}$ or b$)=\mathrm{P}(\mathrm{a})+\mathrm{P}(\mathrm{b})-\mathrm{P}(\mathrm{a}$ and b$)$

Proof of (g):
(A) "a" is equivalent to "( $a$ and $b$ ) or ( $a$ and not $b$ )".

Note also that "a and b" and "a and not b" are incompatible.
(B) "a or $b$ " is equivalent to "b or ( $a$ and not $b$ )".

Note also that "b" and "a and not b" are incompatible.
From (A) we learn... $\quad \mathrm{P}(\mathrm{a}) \quad=\mathrm{P}(\mathrm{a}$ and b$)+\mathrm{P}(\mathrm{a}$ and not b$)$
Rearranging... $\quad \mathrm{P}(\mathrm{a}$ and not b$)=\mathrm{P}(\mathrm{a})-\mathrm{P}(\mathrm{a}$ and b$)$
From (B) we learn... $\quad \mathrm{P}(\mathrm{a}$ or b$)=\mathrm{P}(\mathrm{b})+\quad \mathrm{P}(\mathrm{a}$ and not b$)$
Substituting, we get $\quad \mathrm{P}(\mathrm{a}$ or b$)=\mathrm{P}(\mathrm{a})+\mathrm{P}(\mathrm{b})-\mathrm{P}(\mathrm{a}$ and b$)$

## Quick Applications

What are the chances that the roll of a fair six-sided die lands on a 2 or a 5 ?

If you flip a coin twice, the chances of getting heads twice are $1 / 4$. So what are the chances, if you flip a fair coin twice, of getting at least one heads?

What are the chances that a card drawn at random from a deck of 52 cards is either an ace or a spade?

