Mixed Strategies

Last time I mentioned the following notions.

> A strategy is ***pure*** if it involves no randomization.
> A strategy is ***mixed*** if it involves randomization of pure strategies.

The notion of a mixed strategy is useful for coping with games in which there is no pure strategy Nash equilibrium. Consider, for example, the game of “matching pennies”.

Column

Row		Heads	Tails
	Heads	-1 1	1 -1
	Tails	1 -1	-1 1

One reason Nash equilibria were of interest were because you could play your half of the strategy profile openly, or as if your opponent could predict it, and be confident they would play a strategy that wouldn't make you regret your own strategy. But there's no pure strategy like this here.

Instead of picking one strategy, why not randomize? Suppose you flipped your penny, and let the outcome decide what you played? Let's write this strategy “(.5)Heads;(.5)Tails”. How much utility would this strategy yield against various strategies? We need to appeal to expected utilities of strategy profiles...

Question: If a game has no pure-strategy equilibria, might it have a mixed-strategy equilibrium?

> ***Fact 1.*** *Every game with finitely many moves has at least one Nash equilibrium, if mixed strategies are taken into account.*

Given these things are around, it might be nice to have a method for finding Nash equilibria.

First note:

> ***Fact 2***. *If a mixed strategy over strategies S_1, S_2,...is a best response to A, then S_1 is a best response to A, S_2 is a best response to A,...*

Suppose S_1;S_2;... is the best response to A and S_1 isn't a best response to A, the utility of selecting S_1 as a pure strategy would have to be strictly lower than the utility of the best response to A. But then removing S_1 from the mix would *raise* the expected utility of the mixed strategy. But by hypothesis this is impossible, since S_1;S_2;... is a *best* response to A.

Now this means that the expected utility of a mixed strategy best response is equal to the expected utility of its pure strategies! This gives us a great way of finding mixed strategy Nash equilibria. Take our earlier game. Suppose there is some mixed-strategy equilibrium. Then it would look like this.

<(p)Heads;(1-p)Tails, (q)Heads;(1-q)Tails>

Since (p)Heads;(1-p)Tails is a best response to Column's strategy, so is just playing Heads and Tails. But this means EU_{Row}(Heads) = EU_{Row}(Tails). But

$EU_{Row}(\text{Heads}) = 1(q) + -1(1-q) = 2q - 1$
$EU_{Row}(\text{Tails}) = -1(q) + 1(1-q) = -2q + 1$

If these are equal:

$-2q + 1 = 2q-1$ ...so... $2 = 4q$ ...or... $p=1/2$

Similarly

$EU_{Col}(\text{Heads}) = -1(p) + 1(1-p) = -2p + 1$
$EU_{Col}(\text{Tails}) = 1(p) + -1(1-p) = 2p-1$

So p=1/2 and our Nash Equilibrium is

<(1/2)Heads;(1/2)Tails, (1/2)Heads;(1/2)Tails>

So to find a mixed strategy Nash equilibrium for a two player game:

(1) write out what the Nash equilibrium would look like with variables for the probabilities of the strategies p and q.
(2) Find the expected utility of each of the *pure* strategies for Row in terms of the probabilities in Column's mixed strategy (i.e. in terms q).
(3) Set the two equations from (2) equal to each other and solve with simple arithmetic.
(4) Repeat steps (2)–(3) for Column (with p instead of q).
(5) Plus the values p and q back into the mixed strategy you wrote in (1).